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Common Fixed Point Theorems for Two Selfmaps of a Complete S-Metric Space

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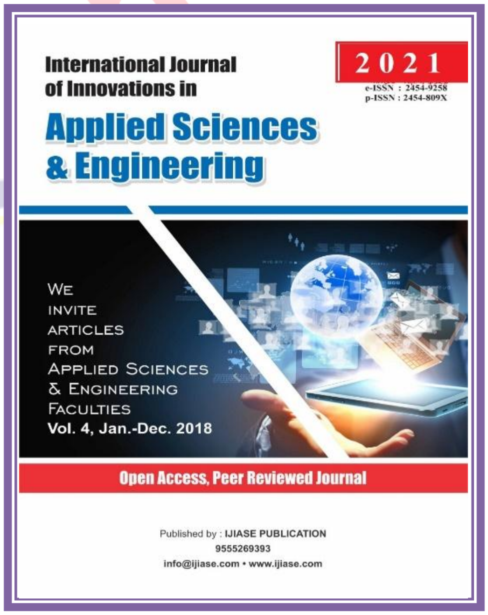

ABSTRACT

The purpose of this paper is to prove a common fixed point theorem for two selfmaps of a complete S-metric space. Also we show that a common fixed point theorem for two selfmaps of a metric space proved by Das and Naik ([5]) follows as a particular case of our result.

Mathematics Subject Classification: 47H10, 54H25.

Key Words: S-metric space; Associated sequence; Fixed point theorem.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory constitutes a fundamental pillar of mathematical analysis, serving as a powerful framework for the systematic study of mappings and their structural properties in a wide range of mathematical spaces. Among the classical results in this area, the Banach fixed point theorem commonly referred to as the Contraction Mapping Theorem holds a distinguished position as perhaps the most celebrated result within the theory of metric spaces. The significance of this theorem transcends its original formulation, as it has inspired a vast body of research devoted to its refinement, extension, and application across diverse mathematical and applied contexts. In particular, the last few decades have witnessed an intensification of efforts aimed at exploring fixed point results within increasingly sophisticated and generalized structures, thereby motivating the development and investigation of fixed point theorems across various classes of metric spaces

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2- metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [7], [8] and [9]. Sedghi et al. [10] modified the concepts of D-metric space and introduced the concepts of D*- metric space also proved a common fixed point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they gives following concepts of S- metric space.

Definition 1.1([11]): Let X be a non-empty set. An S-metric space on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions, for each x, y, z, a ∈ X

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$.

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S–metric space**. Immediate examples of such S-metric spaces are:

Example 1.2: Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X.

Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S- metric on X. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$
$$\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x)$$
$$= S(x, x, a) + S(y, y, a) + S(z, z, a)$$

Example 1. 4: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Remark 1. 5: it is easy to see that every D*-metric is S-metric, but in general the converse is not true, see the following example.

Example 1. 6: Let $X = \mathbb{R}^n$ and $\|\ .\ \|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X, but it is not D*-metric because it is not symmetric.

Lemma 1. 7: In an S–metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S-metric, we get

$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x)$…… (1)

and similarly

$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y)$……(2)

Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition 1.8: Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X; S(x, y, y) < r\}$$

$$B_S[x, r] = \{y \in X; S(x, y, y) \leq r\}$$

For example, Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Therefore $B_S(1, 2) = \{y \in \mathbb{R}\, ; S(y, y, 1) < 2\}$

$$= \{y \in \mathbb{R}\, ; |y - 1| < 1\} = (0, 2).$$

Definition 1.9: Let (X, S) be an S–metric space and $A \subset X$.

(1)If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X

(2) If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S–bounded.**

(3) A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for

each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \in$ and we denote this by $\lim_{n \to \infty} x_n = x$

(4) A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \in$ for each $m, n \geq n_0$

(5) The S–metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

(7) If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S–metric space.

Lemma 1. 10([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball

$B_S(x, r)$ is an open subset of X.

Lemma1. 11([11]**)**: Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x,

then x is unique.

Lemma1. 12([11]**)**: Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x,

then $\{x_n\}$ is a Cauchy sequence.

Lemma1. 13([11]**)**: Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such

that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$,

then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma1. 14: Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all x, y, z ∈ X is an S-metric on X
2. $x_n \to x$ in (X, d) if and only if $X_n \to x$ in (X, S_d)
3. $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)
4. (X, d) is complete if and only if (X, S_d) is complete

Proof: **(1)** See [Example (3), Page 260]

(2) $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \to 0$ that is, $x_n \to x$ in (X, S_d)

(3) $\{x_n\}$is a Cauchy in (X, d) if and only if $d(x_n, x_m) \to 0$ as n, m → ∞, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \to 0$ n, m → ∞, that is, $\{x_n\}$ is Cauchy in (X, S_d)

(4) It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X, we denote T(x) by Tx.

If P and Q are selfmaps of a set X, then any z ∈ X such that Pz = Qz = z is called a **common fixed point** of P and Q.

Two selfmaps P and Q of X are said to be **commutative** if PQ = QP where PQ is their composition PoQ defined by (PoQ) x = PQx for all x ∈ X.

Definition 1.15: Suppose P and Q are selfmaps of a S–metric space (X, S) satisfying the condition $Q(X) \subseteq P(X)$. Then for any $x_0 \in X$, $Qx_0 \in Q(X)$ and hence $Qx_0 \in P(X)$, so that there is a $x_1 \in X$ with $Qx_0 = Px_1$, since $Q(X) \subseteq P(X)$. Now $Qx_1 \in Q(X)$ and hence there is a $x_2 \in X$ with $Qx_2 \in Q(X) \subseteq P(X)$ so that $Qx_1 = Px_2$. Again $Qx_2 \in Q(X)$ and hence $Qx_2 \in P(X)$ with $Qx_2 = Px_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$. We shall call this sequence as an **associated sequence of x_0 relative to the two selfmaps P and Q.** It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps P and Q.

Let P and Q are selfmaps of a S-metric space (X, S) such that $Q(X) \subseteq P(X)$. For any $x_o \in X$, if $\{x_n\}$ is a sequence in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps P and Q.

Definition 1.16: A function Ø: [0, ∞) → [0, ∞) is said to be a **contractive modulus**, if Ø(0) = 0 and Ø(t) < t for t > 0.

Definition 1.17: A real valued function Ø defined on $X \subseteq \mathbb{R}$ is said to be **upper semi continuous**, if $\lim_{n\to\infty} \sup Ø(t_n) \leq Ø(t)$ for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n \to \infty$.

Definition 1.18: If P and Q are selfmaps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = t$, we have

$\lim_{n\to\infty} S(PQx_n, QPx_n, QPx_n) = 0$, then we say that P and Q are **compatible.**

2. THE MAIN RESULTS:

2. 1 Introduction: If (X, S) is a complete S-metric space and P, Q are selfmaps satisfying certain conditions, we shall prove that, they have a common fixed point.

2.1.1 Theorem: Let P and Q be selfmaps of a S-metric space (X, S) satisfying the conditions

(i) $Q(X) \subseteq P(X)$

(ii) $S(Qx, Qy, Qy) \leq \emptyset(\xi(x, y))$ for all x, y ϵ X

where Ø is an upper semi continuous and contractive modulus and

(ii)′ $\xi(x, y) = \max\{S(Px, Py, Py), S(Px, Qx, Qx), S(Py, Qy, Qy),$

$\frac{1}{2}[S(Px, Qy, Qy) + S(Py, Qx, Qx)]\}$

(i) P is continuous and

(iv) the pair (P, Q) is compatible

Further, if

(ii) there exists a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the two selfmaps such that the sequences $\{Px_n\}$and $\{Qx_n\}$ converge to some z ϵ X,

Then z is the unique common fixed point for P and Q.

Proof: From (v), we get

(2.1.2) $Px_{2n}, Qx_{2n}, Px_{2n+1}, Qx_{2n+1} \to z$ as $n \to \infty$.

Now, since P is continuous, we have, by (2.1.2)

(2.1.3) $P^2x_{2n+1} \to Pz$ and $PQx_{2n+1} \to Pz$ as $n \to \infty$

Since the pair (P, Q) is compatible, we have, in view of (2.1.2) that

(2.1.4) $\lim_{n\to\infty} S(PQx_{2n+1}, QPx_{2n+1}, QPx_{2n+1}) = 0$

(2.1.5) $QPx_{2n+1} \to Pz$ as $n \to \infty$.

Also from (ii), we have

(2.1.6) $S(QPx_{2n+1}, Qx_{2n}, Qx_{2n}) \leq \emptyset(\xi(Px_{2n+1}, x_{2n}))$

where $\xi(Px_{2n+1}, x_{2n}) = \max\{S(P^2x_{2n+1}, Px_{2n}, Px_{2n}), S(P^2x_{2n+1}, QPx_{2n+1}, QPx_{2n+1}),$

$S(Px_{2n}, Qx_{2n}, Qx_{2n}), \frac{1}{2}[S(P^2x_{2n+1}, Qx_{2n}, Qx_{2n}) + S(Px_{2n}, QPx_{2n+1}, QPx_{2n+1})]\}$

which on letting n to ∞ and using the continuity of S, gives

$\lim_{n\to\infty} \xi(Px_{2n+1}, x_{2n})$ = max{S(Pz, z, z), S(Pz, Pz, Pz), S(z, z, z), $\frac{1}{2}$[S(Pz, z, z) + S(Pz, z, z)]}

= D*(Pz,z,z)

Therefore letting n to ∞ in (2.1.6), and using the above we get

(2.1.7) S(Pz, z, z) ≤ Ø(S(Pz, z, z)).

Now, if Sz ≠ z, then S(Pz, z, z) > 0 and by the definition of Ø, we get

Ø(S(Pz, z, z)) < D*(Pz, z, z) contradicting (2.1.7)

Thus we have Pz = z.

Now again from (ii) we have

(2.1.8) S(Qz, Qx_{2n}, Qx_{2n}) = max {S(Pz, Px_{2n}, Px_{2n}), S(Pz, Qz, Qz), S(Px_{2n}, Qx_{2n} , Qx_{2n}), $\frac{1}{2}$[S(Pz, Qx_{2n}, Qx_{2n})+ S(Px_{2n} , Qz, Qz)]}

in which on letting n to ∞, using Sz = z, the continuity of S and the condition (v), we get

$\lim_{n\to\infty} \xi(z, x_{2n})$ = max {S(Pz, z, z), S(z, Qz, Qz), S(z, z, z), $\frac{1}{2}$[S(Pz, z, z) + S(Pz ,z, z)]}

= S(z, Qz, Qz).

Again letting n to ∞ in (2.1.8) and using the above, we get

(2.1.9) S(Qz, z, z) ≤ Ø(S(Qz, z, z))

and this will be a contradiction if Qz ≠ z, therefore Qz = z.

Thus 'z' is a common fixed point of P and Q.

To prove that z is the unique common fixed point of P and Q. If possible suppose that z' is another common fixed point of P and Q, then from (ii), we have

(2.1.10) S(z, z', z') = S(Qz, Qz', Qz') ≤ Ø(ξ(z, z')).

where ξ(z, z')= max{S(Pz, z', z'), S(z, Qz, Qz), S(Pz', Qz', Qz'), $\frac{1}{2}$[S(Pz, z', z')+S(Pz', Qz,Qz)]}

= S(z, z', z')

so that (2.1.10) gives

(2.1.11) S(z, z', z') ≤ Ø (S(z, z', z'))

and this will give a contradiction if z ≠ z'. Therefore z = z'. Thus z is the unique common fixed point of P and Q.

2.2 A Common Fixed Point Theorem for Two Selfmaps of a Complete S - metric space

2.2.1 Theorem: Suppose P and Q are selfmaps of a S -metric space (X, S) satisfying conditions (i) to (iv) of Theorem 2.1.1.

Further, if

(v)' (X, S) is complete.

then P and Q have unique common fixed point.

Before we give a proof of this, we prove the following lemma

2.2.2 Lemma: Let (X, S) be a S-metric space and P and Q be selfmaps of X such that

(i) $Q(X) \subseteq P(X)$

(ii) $S(Qx, Qy, Qy) \leq c\, \xi(x, y)$ for all $x, y \in X$

where $0 \leq c < 1$ and $\xi(x, y)$ is as defined in (ii)' of Theorem 2.1.1

Further if

(iii) (X, S) is complete

then for each $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the selfmaps, the sequences $\{Qx_n\}$ and $\{Px_n\}$ converges to same point $z \in X$.

Proof: Suppose P and Q are selfmaps of a S-metric space (X, S) for which the conditions (i) and (ii) hold.

Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to two selfmaps. Then, since $Qx_{2n} = Px_{2n+1}$ and $Qx_{2n+1} = Px_{2n+2}$ for $n \geq 0$.

Note that

$\xi(x_{2n}, x_{2n+1}) = \max \{S(Px_{2n}, Px_{2n+1}, Px_{2n+1}), S(Px_{2n}, Qx_{2n}, Qx_{2n}), S(Px_{2n+1}, Qx_{2n+1}, Qx_{2n+1}),$

$\frac{1}{2}[S(Px_{2n}, Qx_{2n+1}, Qx_{2n+1}) + S(Px_{2n+1}, Qx_{2n}, Qx_{2n})]\}$

$= \max \{S(Px_{2n}, Qx_{2n}, Qx_{2n}), S(Px_{2n}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1}),$

$\frac{1}{2}[S(Px_{2n}, Qx_{2n+1}, Qx_{2n+1}) + S(Qx_{2n}, Qx_{2n}, Qx_{2n})]\}$

$= \max \{S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1}), \frac{1}{2} S(Qx_{2n-1}, Qx_{2n+1}, Qx_{2n+1})\}$.

$\xi(x_{2n}, x_{2n+1}) \leq \max \{ S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1})\}$

since $\frac{1}{2} S(Qx_{2n-1}, Qx_{2n+1}, Qx_{2n+1}) \leq \max \{S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1})\}$

Now, by (ii), $S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1}) \leq c \,.\, \xi(x_{2n}, x_{2n+1}) \leq c. \max \{ S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1})\}$

since $0 \leq c < 1$, it follows that the

$\max \{ S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}), S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1})\} = S(Qx_{2n-1}, Qx_{2n}, Qx_{2n})$

therefore $S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1}) \leq c. S(Qx_{2n-1}, Qx_{2n}, Qx_{2n})$ ……..(A)

Similarly, we can show that

$S(Qx_{2n-1}, Qx_{2n}, Qx_{2n}) \leq c. S(Qx_{2n-1}, Qx_{2n-2}, Qx_{2n-2})$ ……..(B)

From (A) and (B), we get

$S(Qx_{2n}, Qx_{2n+1}, Qx_{2n+1}) \leq c^2 S(Qx_{2n-1}, Qx_{2n-2}, Qx_{2n-2})$

$\leq c^4$ S(Qx_{2n-3}, Qx_{2n-4}, Qx_{2n-4})

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$\leq c^{2n}$ S(Qx_1, Qx_0, Qx_0)

Since $c^{2n} \to 0$ as $n \to \infty$ (because c < 1), the sequence $\{Qx_n\}$ is a Cauchy sequence in (X, S) and since it is complete, it converges to a point say z ϵ X.

Similarly we can prove that $\{Px_{2n}\}$ converges to a point say z' ϵ X. Since $Px_{2n+1} = Qx_{2n}$, we get z = z'. (In fact, z' = $\lim_{n\to\infty} P x_{2n+1} = \lim_{n\to\infty} Q x_{2n}$ = z), proving lemma.

2.2.3 Remark: The converse of Lemma 2.2.2 is not true. That is suppose P and Q are selfmaps of a S-metric space (X, S) satisfying condition (i) and (ii) of Lemma 2.2.2; even if for each x_0 ϵ X and for each associated sequence $\{x_n\}$ of x_0 relative to P and Q, the sequence $\{Px_n\}$ and $\{Qx_n\}$ converges in X, then (X, S) need not complete.

2. 2. 4 Corollary: Suppose P and Q are selfmaps of a S-metric space (X, S) satisfying conditions (i) to (iv) of Theorem 2. 1. 1. Further, if

(v)′ (X, S) is complete

then P and Q have unique common fixed point.

Proof: In view of Lemma 2.2.2 the condition (v) of Theorem 2.1.1 holds in view of (v)' Hence the corollary follows from Theorem 2.1.1.

2.2.5: Remark: Taking Ø(t) = c t where 0 ≤ c < 1 in the Theorem 2.1.1, we get the following corollary immediately.

2.2.6: Corollary: Let P and Q be selfmaps of a S-metric space (X, S) satisfying the conditions (i), (iii), (iv), (v) and

(ii)" S(Qx, Qy, Qy) ≤ c ξ(x, y)

where ξ(x, y) is same as defined in (ii)' of Theorem 2.1.1. Then z is the unique common fixed point of P and Q.

Now we show that a common fixed point theorem for two selfmaps of metric space proved by Das and Naik ([5]) follows as a particular case of our Theorem.

2.2.7 Corollary ([5]): Let P and Q be two selfmaps of a metric space (X, d) such that

(i) Q(X) ⊆ P(X)

(ii) d(Qx, Qy) ≤ Ø(η(x, y)) for all x, y ϵ X,

where 0 ≤ c < 1 and

(ii)' η(x, y) = max {d(Px, Py), d(Px, Qx), d(Py, Qy), d(Px, Qy), d(Py, Qx)}

(iii) P is continuous,

and

(iv) PQ = QP

Further, if

(v) X is complete

Then P and Q have a unique common fixed point in X.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If S(x, y, z) = max {d(x, y), d(y, z), d (z, x)} then (X, S) is a S-metric space and S(x, y, x)= d(x, y). Therefore (ii) can be written as S(Qx, Qy, Qy) ≤ c. η(x, y) for all x, y ϵ X where η(x, y) = max {S(Px, Py, Py), S(Px, Qx, Qx), S(Py, Qy, Qy), S(Px, Qy, Qy), S(Py, Qx, Qx)} = ξ(x, y),

which is the same as condition (ii) of Theorem 2.2.1. Also since (X, d) is complete, we have (X, S) is complete by Corollary1.13.

Now, P and Q are selfmaps on (X, S) satisfying conditions of Theorem 2.2.1 and hence the corollary follows.

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